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Variational formulation and optimal control of fractional diffusion equations with Caputo derivatives

Qing Tang[1*] and Qingxia Ma[1,2]

*Correspondence:
tangqingthomas@gmail.com
[1]School of Mathematics and Physics, China University of Geosciences, Lumo 388, Wuhan, 430074, China
Full list of author information is available at the end of the article

Abstract
In this paper we start by giving a new definition of weak Caputo derivative in the sense of distributions, and we give a variational formulation to a fractional diffusion equation with Caputo derivative. We prove the existence and uniqueness of the solution to this weak formulation and use it to obtain a result on optimal control.

MSC: 37L05; 47J30

Keywords: fractional diffusion equation; distributional weak Caputo derivative; fractional Sobolev space

1 Introduction

In this paper we study fractional diffusion equations with controls by the method of an abstract variational formulation. For a comprehensive treatment of the subject of fractional calculus and fractional differential equations we refer to Kilba *et al.* [1]. There has been a large and fast increasing literature on diffusion equations with time fractional derivatives [2–7]. An important obstacle to study solutions in fractional Sobolev spaces is that the Caputo derivative was not clearly defined when the first order derivative does not exist in the strong sense. In the recent work of Gorenflo *et al.* [5], they gave a definition of the Caputo derivative in fractional Sobolev space. In this paper we attempt to give a new definition of the weak fractional Caputo derivative via distribution theory and an integration by parts formula. This definition makes it very natural to adopt the theory of operational differential equations (Lions [8]) and gives an abstract variational formulation of the fractional diffusion equation.

We will study this weak formulation with the classic Faedo-Galerkin method. This is essentially a Hilbert space method [9]. We adopt it for fractional diffusion equations by using energy estimate inequalities and lemmas on weak convergence.

We note that the integration by parts technique has been developed and extensively used in the theory of fractional calculus of variations, of which we refer to the monograph of Malinowska and Torres [10].

We obtain a solution in the functional setting of a fractional Sobolev space due to the following fact: the L^2-fractional derivative is the fractional power of the realization of a derivative in L^2 space [11]. For a detailed analysis and characterization of the fractional power of differential operator in the setting of Sobolev space, we refer to [11, 12].

Using the fractional integration by parts formula, we can also construct the adjoint system to our variational (weak) formulation. By a classic result of convex analysis we can characterize the optimal control of a system of partial differential equations and inequalities, which can be applied to concrete fractional diffusion equations.

Consider the following fractional diffusion equation: Let Ω be a bounded open subset of $\mathbb{R}^n$ with sufficiently regular boundary Γ, with cylinder defined as

$$Q = \Omega \times]0, T[, \qquad \Sigma = \Gamma \times]0, T[.$$

We will study the system with Dirichlet boundary condition:

$$\begin{cases} \partial_t^\alpha y - \sum_{i,j=1}^n \frac{\partial}{\partial x_i}(a_{ij}(x,t)\frac{\partial y}{\partial x_j}) = f & \text{in } Q, t \in (0,T], \\ y = 0 & \text{on } \Sigma, \\ y(x,0) = 0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

and the system with distributed control:

$$\begin{cases} \partial_t^\alpha y(u) - \sum_{i,j=1}^n \frac{\partial}{\partial x_i}(a_{ij}(x,t)\frac{\partial y(u)}{\partial x_j}) = f + u & \text{in } Q, t \in (0,T], \\ y = 0 & \text{on } \Sigma, \\ y(x,0) = 0 & \text{in } \Omega, \end{cases} \tag{1.2}$$

here $y(t,x;u)$ and u denote the state and control, respectively, $\alpha \in (0,1)$.

We assume the conditions for the coefficient:

$$a_{ij} \in L^\infty(Q) \text{ and symmetric}, \quad 1 \le i,j \le n. \tag{1.3}$$

Also $\exists \theta > 0$, such that $\forall \xi \in \mathbb{R}^n$,

$$\sum_{i,j=1}^n a_{ij}(x,t)\xi_i\xi_j \ge \theta \sum_{i=1}^n \xi_i^2, \quad \text{a.e. in } Q. \tag{1.4}$$

Our approach has the following novelties:

1. We define the notion of a weak Caputo derivative and prove that it has similar properties to the weak derivative defined in [5] (norm equivalence with the inverse of Riemann-Liouville operator).
2. The variational method allows us to consider fractional diffusion equation where the coefficient $a_{ij}(x,t)$ of the diffusion operator depends on time.
3. The variational method allows us to consider f taken in a larger space than $L^2(Q)$.
4. Our weak formulation provides the start of a theory for control problems of fractional diffusion equations, which is natural and analogous to control theory of partial differential equations (with time derivative of integer order).

2 Preliminaries

H is a separable Hilbert space. V is a dense subspace of H with continuous injection:

$$V \hookrightarrow H;$$

as usual, H is identified with its dual H'. We denote the evolution triple $\{V, H, V'\}$ with embeddings:

$$V \hookrightarrow H \hookrightarrow V' \tag{2.1}$$

being continuous and dense.

We introduce the vector valued space $L^2(0,T;V)$ such that $y \in L^2(0,T;V)$ if

$$\left(\int_0^T \|y(t)\|_V^2\,dt\right)^{1/2} < \infty.$$

In the same way we can define $L^2(0,T;H)$, $L^2(0,T;V')$. We have the following.

Let E_1, E_2 be two Banach spaces, $\mathcal{L}(E_1,E_2)$ denotes the space of continuous linear mapping of $E_1 \to E_2$. $A(t)$ is an unbounded operator in H, and continuous linear operator in V itself, such that

$$A \in \mathcal{L}\big(L^2(0,T;V), L^2\big(0,T;V'\big)\big), \qquad A(t) \in \mathcal{L}\big(V, V'\big).$$

We denote $(A(t)y(t), v) = a(t; y(t), v)$ for all $v \in V$, and give the following conditions:

(A1) $a(t; y(t), v) \leq \mu\|y(t)\|\|v\|$, where μ is a constant independent of $t \in [0,T]$, $y, v \in V$.

(A2) $t \to a(t; y(t), v)$ is measurable in $]0,T[$ for all $y(t), v \in V$.

(A3) $\exists\theta > 0$ independent of t, $\operatorname{Re} a(t; v, v) \geq \theta\|y\|^2$, $\forall v \in V$. Hence $a(t; y(t), v)$ is coercive uniformly in t.

To avoid confusion, we clarify here that we use Dom or D to denote the domain of an operator, and $\mathcal{R}$ denotes the range.

Definition 2.1 The left and right Caputo derivative are defined, for $0 < t \leq T$, $0 < \alpha < 1$, by

$$\partial_t^\alpha y(t) = {}_0^C D_t^\alpha y(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\frac{\partial y}{\partial s}(s)\,ds,$$

$${}_t^C D_T^\alpha y(t) = \frac{1}{\Gamma(1-\alpha)}\int_t^T (s-t)^{-\alpha}\frac{\partial y}{\partial s}(s)\,ds.$$

From this definition we see that for $\partial_t^\alpha y(t)$ to be well defined, $\partial y/\partial t$ must be well defined. This is quite restrictive in applications, hence motivating our study of the weak Caputo derivative.

Next we define left and right Riemann-Liouville integrals, for $0 \leq t \leq T$, $0 < \alpha \leq 1$:

$$\big(J^\alpha y\big)(t) = {}_0 I_t^\alpha y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} y(s)\,ds, \quad J^0 = I,$$

$${}_t I_T^\alpha y(t) = \frac{1}{\Gamma(\alpha)}\int_t^T (s-t)^{\alpha-1} y(s)\,ds.$$

When $y \in L^1(0,T)$, the left Riemann-Liouville integral can also be defined via a convolution ((1.7), p.10, [7]):

$$(J^\alpha y)(t) = (g^\alpha * y)(t),$$

where

$$g^\alpha(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} t^{\alpha-1}, & t > 0, \\ 0, & t \leq 0. \end{cases}$$

We denote by $H^\alpha(0,T)$ the fractional Sobolev space of order α on $(0,T)$.

$${}_0H^\alpha(0,T) = \{y \in H^\alpha(0,T) : y(0) = 0\}.$$

For details of these definitions we refer to Section 9.1, Chapter 1 of [13].

It has been verified in previous literature that the Riemann-Liouville integral operator is injective [5], hence we can define a new operator $J^{-\alpha}$ as the inverse operator of J^α. By definition, $D(J^{-\alpha}) = \mathcal{R}(J^\alpha)$.

We use the following result regarding a solution to a fractional differential equation [5]:

$$\partial_t^\alpha y = f, \quad f \in L^2,$$

is given by $u = J^\alpha f$ with the range of this operator J^α being

$$\mathcal{R}(J^\alpha) = \begin{cases} H^\alpha(0,T), & 0 \leq \alpha < \frac{1}{2}, \\ {}_0H^\alpha(0,T), & \frac{1}{2} < \alpha \leq 1, \\ \{y \in H^{\frac{1}{2}}(0,T) : \int_0^T t^{-1}|y(t)|^2\,dt < \infty\}, & \alpha = \frac{1}{2}. \end{cases} \tag{2.2}$$

In the literature of interpolation theory, one sometimes denotes (Remark 11.4, [13])

$$[{}_0H^1(0,T), H^0(0,T)]_{\frac{1}{2}} = {}_0H_0^{\frac{1}{2}}(0,T) = \{y \in H^{\frac{1}{2}}(0,T) : t^{\frac{1}{2}} y \in L^2(0,T)\}.$$

With the above definitions we have the following result.

Lemma 2.2 (Theorem 2.1, [5]) *The norms $\|J^{-\alpha}y\|_{L^2}$ and $\|y\|_{H^\alpha(0,T)}$ are equivalent for $y \in \mathcal{R}(J^\alpha)$.*

We denote by $H^\alpha(0,T;V,V')$ the vector valued fractional Sobolev space.

$W^{\alpha,p}(0,T;V,V')$ is the restriction on $(0,T)$ of $W^{\alpha,p}(-\infty,\infty;V,V')$ given by the Fourier transform and

$$H^\alpha(0,T;V,V') = W^{\alpha,2}(0,T;V,V').$$

Next we give a lemma from Simon [14], which gives the embedding between fractional Sobolev spaces and spaces of continuous functions, for vector spaces on an interval.

Lemma 2.3 ((6.5), [15]) *Denote by $C_u(0,T;H)$ the space of uniformly continuous functions from $(0,T)$ into H. Suppose $\alpha > 1/p$ $(0 < \alpha < 1, 1 < p \leq \infty)$. Then*

$$W^{\alpha,p}(0,T;V,V') \hookrightarrow C_u(0,T;H) \quad \textit{with compact embedding.}$$

Lemma 2.4 ((6.5), [15]) *Let α, p and q satisfy*:
if $\alpha > 1/p$ then $p \leq q \leq \infty$,
if $\alpha = 1/p$ then $p \leq q < \infty$,
if $\alpha < 1/p$ then $p \leq q \leq p_*$, where $\alpha - 1/p = -1/p_*$, that is, $p_* = p/(1-\alpha p)$ $(0 < \alpha < 1$, $1 \leq p \leq q \leq \infty)$.
Then

$$W^{\alpha,p}(0,T;V,V') \hookrightarrow L^q(0,T;H) \quad \textit{with compact embedding.} \tag{2.3}$$

In order to use the theory of operational differential equations, we need to interpret the weak Caputo derivative in the sense of distributions, through fractional integration by parts in the formula ((1) in [16])

$$\int_0^T \big(\partial_t^\alpha y(t), \psi(t)\big)\,dt = \int_0^T \big(y, {}_t^C D_T^\alpha \psi(t)\big)\,dt + \big[{}_t I_T^{1-\alpha} \psi(t) \cdot y(t)\big]_0^T. \tag{2.4}$$

For $y(0) = 0$, $\psi(T) = 0$, we have

$$\big[{}_t I_T^{1-\alpha} \psi(t) \cdot y(t)\big]_0^T = 0.$$

Hence we can proceed to the construction of a weak Caputo derivative in the sense of distributions (Schwartz [17]), note that if a distribution function is infinitely differentiable then its Caputo fractional derivative must also exist. It greatly simplifies the situation since we have the initial condition $y(0) = y_0 = 0$.

We denote by $\mathcal{D}(]0,T[)$ the space of infinitely differentiable functions in $]0,T[$ with compact support. We call every continuous linear mapping of $\mathcal{D}(]0,T[)$ into E a vectorial distribution over $]0,T[$ with values in a Banach space E, and we denote

$$\mathcal{D}'\big(]0,T[;E\big) = \mathcal{L}\big(\mathcal{D}(]0,T[);E\big).$$

Definition 2.5 Define the test function $\varphi \in \mathcal{D}(]0,T[)$ for the function y such that $y(0) = 0$, we call $\partial_t^\alpha y$ a distributional weak Caputo derivative if it is a linear functional on $\mathcal{D}(]0,T[)$ that sends φ into $\int_0^T (y, {}_t^C D_T^\alpha \varphi(t))$.

Our new definition of a weak Caputo derivative generalizes the (left) Caputo derivative (Definition 2.1) since it is well defined even when $\partial y/\partial s$ does not exist in the strong sense. It coincides with the Caputo derivative if $\partial y/\partial s$ does exist.

Lemma 2.6 $\partial_t^\alpha (y(\cdot), v) = \langle \partial_t^\alpha y(\cdot), v\rangle$ *in $\mathcal{D}'(]0,T[)$, for $y \in {}_0H^\alpha(0,T;V,V')$, $v \in V$. Here $(\ ,\ )$ denotes duality in H, $\langle\ ,\ \rangle$ denotes a duality pairing of V' and V. Moreover, the weak Caputo derivative $\partial_t^\alpha y = J^{-\alpha} y$ in L^2 for $y \in \mathcal{R}(J^\alpha)$.*

Proof Denote the function $\varphi \in \mathcal{D}(]0,T[)$. For all t, $\varphi(t)$ is a scalar. We can write $v(t) = \varphi(t)v$. Observe $y(t), v \in V \subset H$ and the duality $\langle\ ,\ \rangle$ is compatible with the identification of H with its dual. This implies

$$\langle v, y(t)\rangle = \bigl(v, y(t)\bigr) = \bigl(y(t), v\bigr).$$

From Definition 2.5 and (2.4) we obtain

$$\begin{aligned}\int_0^T \langle \partial_t^\alpha y(t), v\rangle \varphi(t)\, dt &= \int_0^T \langle v, y(t)\rangle {}_t^C D_T^\alpha \varphi(t)\, dt \\ &= \int_0^T \bigl(y(t), v\bigr) {}_t^C D_T^\alpha \varphi(t)\, dt \\ &= \int_0^T \partial_t^\alpha \bigl(y(t), v\bigr)\varphi(t)\, dt,\end{aligned}$$

hence $\partial_t^\alpha (y(\cdot), v) = \langle \partial_t^\alpha y(\cdot), v\rangle$ in $\mathcal{D}'(]0,T[)$.

Since the Sobolev space ${}_0H^2(0,T)$ is dense in $\mathcal{R}(J^\alpha)$, for each $y \in \mathcal{R}(J^\alpha)$ we can construct an approximating sequence ϕ_n such that

$$\lim_{n\to\infty} \phi_n = y, \qquad \phi_n \in_0 H^2(0,T).$$

By the Hahn-Banach theorem we can uniquely extend the domain of linear operator $y \mapsto \partial_t^\alpha y$ from ${}_0H^2(0,T)$ to $\mathcal{R}(J^\alpha)$. From [5] (Lemma 3.1) we know

$$\partial_t^\alpha \phi_n = J^{-\alpha}\phi_n, \qquad \phi_n \in_0 H^2(0,T),$$

hence we obtain

$$\int_0^T \bigl(\partial_t^\alpha y, \varphi\bigr)\, dt = \int_0^T \bigl(J^{-\alpha} y, \varphi\bigr)\, dt, \quad y \in \mathcal{R}\bigl(J^\alpha\bigr).$$

From Definition 2.5 and the fact that $\varphi \in \mathcal{D}(]0,T[) \subset L^2(0,T)$ we obtain the weak Caputo derivative $\partial_t^\alpha y = J^{-\alpha} y$ in L^2 for $y \in \mathcal{R}(J^\alpha)$. □

From Lemma 2.2 and Lemma 2.6 we obtain the following: suppose we have a sequence of approximating solutions $y_m \in \mathcal{R}(J^\alpha)$; if we have a priori estimates independent of m, such that $y_m(t) \in L^2(0,T;V)$ and $\partial_t^\alpha y_m(t) \in L^2(0,T;V')$, then we have $y_m(t) \in H^\alpha(0,T;V,V')$.

Definition 2.7 We define the variational fractional equation (we also call it a fractional operational differential equation). Suppose $f \in L^2(0,T;V')$,

$$\begin{cases} \partial_t^\alpha (y(t), v) + a(t; y(t), v) = \langle f(t), v\rangle & \text{in } \mathcal{D}'(]0,T[), t \in (0,T], \\ \forall v \in V, \\ y_0 = 0. \end{cases} \tag{2.5}$$

Here $\partial_t^\alpha y$ is defined in the weak sense (Definition 2.5).

From Lemma 2.6 we see that the first equation of (2.5) is equivalent to

$$\partial_t^\alpha y + A(t)y = f \quad \text{in the sense of } L^2(0,T;V'), t \in (0,T].$$

Definition 2.8 y is a (distributional) weak solution to system (1.1); it satisfies (2.5) with

$$y \in \begin{cases} H^\alpha(0,T;V,V'), & 0 \le \alpha < \frac{1}{2}, \\ {}_0H^\alpha(0,T;V,V'), & \frac{1}{2} < \alpha \le 1, \\ \{u \in H^{\frac{1}{2}}(0,T;V,V') : \int_0^T t^{-1}|y(t)|^2\,dt < \infty\}, & \alpha = \frac{1}{2}, \end{cases} \tag{2.6}$$

and $V = H_0^1(\Omega)$.

3 Main results

3.1 Existence and uniqueness of solution to the variational formulation

Theorem 3.1 *For $y(0) = 0, f \in L^2(0,T;V')$, conditions* (A1) *to* (A3) *are satisfied, suppose* (2.5) *has a solution $y \in L^2(0,T;V)$, then it is unique.*

Proof Uniqueness. Suppose there exist two different solutions y_1 and y_2, $y_3 = y_1 - y_2$, then

$$\partial_t^\alpha y_3 + A(t)y_3 = 0,$$
$$\int_0^T \bigl(\partial_t^\alpha y_3(t), y_3(t)\bigr)\,dt + \int_0^T a\bigl(t; y_3(t), y_3(t)\bigr)\,dt = 0.$$

Suppose $y_3(t) \in V \subset H$, we know the following inequality (inequality (5.21), [7]):

$$\int_0^T \left(\frac{d}{dt}\bigl(g^{1-\alpha} * y_3(t)\bigr), y_3(t)\right)_H dt \ge g^{1-\alpha}(T)\int_0^T \|y_3(t)\|_H^2\,dt. \tag{3.1}$$

Since y is a weak solution hence by Definition 2.1 and Lemma 2.6 we have

$$\partial_t^\alpha y_3(t) = \frac{d}{dt}\bigl(J^{1-\alpha}y_3(t)\bigr) = \frac{d}{dt}\bigl(g^{1-\alpha} * y_3(t)\bigr).$$

Hence we have

$$\int_0^T \bigl(\partial_t^\alpha y_3(t), y_3(t)\bigr)\,dt \ge g^{1-\alpha}(T)\int_0^T \|y_3(t)\|_H^2\,dt,$$

and from condition (A3) we obtain

$$g^{1-\alpha}(T)\int_0^T \|y_3(t)\|_H^2\,dt + \theta\int_0^T \|y_3(t)\|_V^2\,dt \le 0,$$

hence $\|y_3(t)\|_V = 0$ and the solution to (2.5) is unique.

Existence. We proceed by constructing a Galerkin approximation of the equation. Recall that V is a separable Hilbert space. Let $\{V_m\}$ be a family of finite dimensional spaces, and denote

$$d_m = \dim V_m, \qquad \{U_{jm}\}, j = 1,\dots,d_m, \text{ a basis of } V_m.$$

From this we derive a finite dimensional approximation $y_m(t) = \sum_{j=1}^{d_n} b_{jm}(t)U_{jm}$:

$$\begin{cases} (\partial_t^\alpha y_m(t), U_{jm}) + a(t; y_m(t), U_{jm}) = (f(t), U_{jm}), \quad 1 \le j \le d_m, \\ y_m(0) = 0. \end{cases} \tag{3.2}$$

The domain of u_m is

$$\operatorname{Dom}\bigl(J^{-\alpha}\bigr) = \begin{cases} H^\alpha(0, T; V_m), & 0 \le \alpha < \frac{1}{2}, \\ {}_0H^\alpha(0, T; V_m), & \frac{1}{2} < \alpha \le 1, \\ \{u \in H^{\frac{1}{2}}(0, T; V_m) : \int_0^T t^{-1}|y(t)|^2\, dt < \infty\}, & \alpha = \frac{1}{2}. \end{cases} \tag{3.3}$$

We proceed to establish a prior estimate independent of m; by the fact $f \in L^2(I; V')$, using again inequality (3.1) and condition (A3) we obtain

$$g^{1-\alpha}(T)\int_0^T \|y_m(t)\|_H^2\, dt + \theta\int_0^T \|y_m(t)\|_V^2\, dt \le \frac{\theta}{2}\int_0^T \|y_m(t)\|_V^2\, dt + \frac{1}{2\theta}\int_0^T \|f(t)\|_{V'}^2\, dt,$$
$$g^{1-\alpha}(T)\int_0^T \|y_m(t)\|_H^2\, dt + \frac{\theta}{2}\int_0^T \|y_m(t)\|_V^2\, dt \le \frac{1}{2\theta}\int_0^T \|f(t)\|_{V'}^2\, dt,$$

hence we obtain $y_m \in L^2(0, T; V)$,

$\{y_m\}$ is a bounded sequence in $L^2(0, T; V)$.

By the condition on the unbounded linear operator we obtain

$\{Au_m\}$ is a bounded sequence in $L^2\bigl(0, T; V'\bigr)$.

Since $f \in L^2(0, T; V')$ from the form of the equation

$$\partial_t^\alpha y_m = f - A(t)y_m,$$

we have $\partial_t^\alpha y_m \in L^2(0, T; V')$. From Lemma 2.2 and Lemma 2.6 and the boundedness of $\{y_m\}$ we obtain $y_m \in H^\alpha(0, T; V, V')$.

By definition V is a reflexive Banach space (naturally V' as well), then the unit balls of $L^2(0, T; V)$ and $L^2(0, T; V')$ are weakly compact, hence we obtain

$$\begin{cases} \text{(i)} \quad y_m \rightharpoonup y_i & \text{weakly in } L^2(0, T; V), \\ \text{(ii)} \quad Ay_m \rightharpoonup Ay_i & \text{weakly in } L^2(0, T; V'). \end{cases} \tag{3.4}$$

Consider $\varphi \in \mathcal{D}(]0, T[)$ and $v \in \mathcal{V}$, there exists a sequence $\{v_m\}$, $v_m \in V_m$, such that $v_m \to v$ strongly in V. Define

$$\begin{cases} \psi_m = \varphi \otimes v_m \quad (\textit{i.e. } \psi_m(t) = \varphi(t)v_m), \\ \psi = \varphi \otimes v. \end{cases}$$

By the formula of the derivation of a tensor product [17],

$$\frac{d\psi_m(t)}{dt} = \frac{d\varphi(t)}{dt} \otimes v_m, \qquad \frac{d\psi(t)}{dt} = \frac{d\varphi(t)}{dt} \otimes v,$$

and Definition 2.1 we obtain

$$\begin{aligned} {}_t^C D_T^\alpha \psi(t) &= \frac{1}{\Gamma(1-\alpha)} \int_t^T (s-t)^{-\alpha} \frac{\partial \psi}{\partial s}(s)\, ds \\ &= \frac{1}{\Gamma(1-\alpha)} \int_t^T (s-t)^{-\alpha} \frac{\partial \varphi}{\partial s}(s) v\, ds = {}_t^C D_T^\alpha \varphi(t) \otimes v \end{aligned}$$

and

$${}_t^C D_T^\alpha \psi_m(t) = {}_t^C D_T^\alpha \varphi(t) \otimes v_m.$$

Hence ${}_t^C D_T^\alpha \psi_m(t) \to {}_t^C D_T^\alpha \psi(t)$ strongly in V, and we obtain

$$\begin{cases} \text{(i)} \quad \psi_m \to \psi & \text{in } L^2(0,T;V) \text{ strongly,} \\ \text{(ii)} \quad {}_t^C D_T^\alpha \psi_m \to {}_t^C D_T^\alpha \psi & \text{in } L^2(0,T;H) \text{ strongly.} \end{cases} \tag{3.5}$$

Consider the approximating equation ($\langle\ ,\ \rangle$ denotes the duality of V' and V, $\langle\ ,\ \rangle_{V'}$ denotes the duality of V and V')

$$\int_0^T \bigl(y_m(t), {}_t^C D_T^\alpha \psi_m(t)\bigr)\, dt + \int_0^T a\bigl(t; y_m(t), \psi_m(t)\bigr)\, dt = \int_0^T \bigl\langle f(t), \psi_m(t)\bigr\rangle\, dt.$$

We know, for $m \to \infty$, from (3.5(i)):

$$\int_0^T \bigl\langle f(t), \psi_m(t)\bigr\rangle\, dt \to \int_0^T \bigl\langle f(t), \psi(t)\bigr\rangle\, dt,$$

from (3.4(i)) and (3.5(ii)):

$$\int_0^T \bigl(y_m(t), {}_t^C D_T^\alpha \psi_m(t)\bigr)\, dt \to \int_0^T \bigl\langle y(t), {}_t^C D_T^\alpha \psi(t)\bigr\rangle_{V'}\, dt,$$

from (3.4(ii)) and (3.5(i)):

$$\int_0^T a\bigl(t; y_m(t), \psi_m(t)\bigr)\, dt \to \int_0^T a\bigl(t; y(t), \psi(t)\bigr)\, dt.$$

Then we can pass to the limit and obtain

$$\begin{aligned} &\int_0^T \bigl(y(t), v\bigr) {}_t^C D_T^\alpha \varphi(t)\, dt + \int_0^T a\bigl(t; y(t), v\bigr)\varphi(t)\, dt = \int_0^T \bigl\langle f(t), v\bigr\rangle \varphi(t)\, dt, \\ &\int_0^T \bigl(y(t), v\bigr) {}_t^C D_T^\alpha \varphi(t)\, dt = \int_0^T \bigl\langle f(t), v\bigr\rangle \varphi(t)\, dt - \int_0^T a\bigl(t; y(t), v\bigr)\varphi(t)\, dt \\ &\qquad = \int_0^T \bigl\langle f(t) - A(t)y(t), v\bigr\rangle \varphi(t)\, dt. \end{aligned} \tag{3.6}$$

For $\alpha \in (\frac{1}{2}, 1)$, $y \in H^\alpha(0, T; V, V')$ is (apart from a set of measure zero) equal to a continuous function from $[0, T] \to H$.

Finally we have to verify the initial condition $y_0 = 0$ for $\alpha \in (\frac{1}{2}, 1)$: since $y(0) = 0$

$$\int_0^T \langle \partial_t^\alpha y(t), \varphi(t)v \rangle \, dt = \int_0^T \big(y(t), v\big) {}_t^C D_T^\alpha \varphi(t) \, dt$$

and

$$\int_0^T \big(\partial_t^\alpha y_m(t), v_m\big)\varphi(t) \, dt = \int_0^T \big(y_m(t), v_m\big) {}_t^C D_T^\alpha \varphi(t) \, dt + (y_{0m}, v)_0 I_T^{1-\alpha} \varphi(0).$$

Passing to the limit, we have as $m \to \infty$

$$\int_0^T \big(\partial_t^\alpha y_m(t), v_m\big)\varphi(t) \, dt = \int_0^T \langle f(t), v \rangle \varphi(t) \, dt - \int_0^T a(t; y, v)\varphi(t) \, dt, \tag{3.7}$$

$$\int_0^T \big(\partial_t^\alpha y_m(t), v_m\big)\varphi(t) \, dt = \int_0^T \big(y(t), v\big) {}_t^C D_T^\alpha \varphi(t) \, dt + (y_0, v)_0 I_T^{1-\alpha} \varphi(0). \tag{3.8}$$

From (3.6), (3.7), and (3.8) we obtain

$$(y_0, v)_0 I_T^{1-\alpha} \varphi(0) = 0, \quad \forall v \in V,$$

so that $y_0 = 0$. □

3.2 Optimal control of variational formulation

Denote by $\mathcal{U}$ the Hilbert space of controls. We define the control operator $B \in \mathcal{L}(\mathcal{U}; L^2(0, T; V'))$. Consider the system

$$\begin{cases} \partial_t^\alpha y(u) + A(t)y(u) = f + Bu & \text{in } \mathcal{D}'(]0, T[), t \in (0, T], f \in L^2(0, T; V'), \\ y_0(u) = 0, \\ y(u) \in L^2(0, T; V). \end{cases} \tag{3.9}$$

Define observation $Cy(u)$ such that $C \in \mathcal{L}(L^2(0, T; V'); \mathcal{H})$. $\mathcal{H}$ is a Hilbert space. N is given as $N \in \mathcal{L}(\mathcal{U}; \mathcal{U})$ and

$$(Nu, u)_{\mathcal{U}} \geq \nu \|u\|_{\mathcal{U}}^2, \quad \nu > 0. \tag{3.10}$$

Define the cost function:

$$J(u) = \big\| Cy(u) - z_d \big\|_{\mathcal{H}}^2 + (Nu, u)_{\mathcal{U}}. \tag{3.11}$$

$\mathcal{U}_{ad}$ is a closed convex subset of $\mathcal{U}$ (set of admissible controls), $z_d \in \mathcal{H}$. The optimal control problem is to find $w \in \mathcal{U}_{ad}$ such that

$$J(w) = \inf_{u \in \mathcal{U}} J(u).$$

From the fact that the affine map $u \to y(u)$ is continuous and (3.10), by Theorem 1.1 of [18], there exists a unique optimal control $w \in \mathcal{U}_{ad}$ for (3.9) with cost function given as (3.11).

Lemma 3.2 (Lemma 1.4, [18]) *$w \in \mathcal{U}_{ad}$ is optimal if and only if*

$$J'(w) \cdot (u - w) \geq 0, \quad \forall u \in \mathcal{U}_{ad}. \tag{3.12}$$

From (3.9) and (3.11) we see that (3.12) is equivalent to

$$\bigl\langle Cy(w) - z_d, C\bigl(y(u) - y(w)\bigr)\bigr\rangle_{\mathcal{H}} + \langle Nw, u - w\rangle_{\mathcal{U}} \geq 0, \quad \forall u \in \mathcal{U}_{ad}. \tag{3.13}$$

Denote by Λ (and $\Lambda_{\mathcal{U}}$) the canonical isomorphism of $\mathcal{H}$ onto $\mathcal{H}'$ (and of $\mathcal{U}$ onto $\mathcal{U}'$). Then it reduces to

$$\bigl\langle C^* \Lambda\bigl(y(w) - z_d\bigr), y(u) - y(w)\bigr\rangle + \langle Nw, u - w\rangle_{\mathcal{U}} \geq 0, \quad \forall u \in \mathcal{U}_{ad}. \tag{3.14}$$

We introduce the adjoint system:

$$\begin{cases} {}_t^C D_T^{\alpha} p(u) + A^*(t) y(u) = C^* \Lambda(Cy(u) - z_d), \quad t \in (0, T], \\ p(T; u) = 0, \\ p(u) \in L^2(0, T; V). \end{cases} \tag{3.15}$$

Theorem 3.3 *The optimal control $w \in \mathcal{U}$ is characterized by systems of partial differential systems and inequality*:

$$\begin{cases} \partial_t^{\alpha} y(w) + A(t) y(w) = f + Bw \quad \textit{in } \mathcal{D}'(]0, T[), t \in (0, T], \\ y(0; w) = 0, \end{cases} \tag{3.16}$$

$$\begin{cases} {}_t^C D_T^{\alpha} p(w) + A^*(t) y(w) = C^* \Lambda(Cy(w) - z_d), \quad t \in (0, T], \\ p(T; w) = 0, \end{cases} \tag{3.17}$$

$$\bigl(\Lambda_{\mathcal{U}}^{-1} B^* p(w) + Nw, u - w\bigr)_{\mathcal{U}} \geq 0, \quad \forall u, w \in \mathcal{U}_{ad}, \tag{3.18}$$

$$\begin{cases} y(w) \in L^2(0, T; V), \\ p(w) \in L^2(0, T; V). \end{cases} \tag{3.19}$$

Here $(\, , \,)_{\mathcal{U}}$ denotes the scalar product between $\mathcal{U}'$ and $\mathcal{U}$.

Proof From $C^* \in \mathcal{L}(H'; L^2(0, T; V'))$ we see that (3.13) is equivalent to

$$\int_0^T \bigl\langle C^* \Lambda\bigl(y(w) - z_d\bigr), y(u) - y(w)\bigr\rangle \, dt + \langle Nw, u - w\rangle_{\mathcal{U}} \geq 0, \quad \forall u \in \mathcal{U}_{ad}. \tag{3.20}$$

By Definition 2.5 and the definition of an adjoint operator we obtain

$$\int_0^T \bigl\langle {}_t^C D_T^{\alpha} p(w), y(u) - y(w)\bigr\rangle \, dt = \int_0^T \bigl\langle p(w), \partial_t^{\alpha} y(u) - \partial_t^{\alpha} y(w)\bigr\rangle \, dt, \tag{3.21}$$

$$\int_0^T \bigl\langle A(t)^* p(w), y(u) - y(w)\bigr\rangle \, dt = \int_0^T \bigl\langle p(w), A(t) y(u) - A(t) y(w)\bigr\rangle \, dt. \tag{3.22}$$

By adding (3.18) and (3.19), and using the first equation of (3.15), we obtain

$$\begin{aligned}
&\int_0^T \big\langle C(t)^*\Lambda\big(Cy(w)-z_d\big), y(u)-y(w)\big\rangle\,dt \\
&\quad= \int_0^T \big\langle p(w), \big(A(t)+\partial_t^\alpha\big)\big(y(u)-y(w)\big)\big\rangle\,dt \\
&\quad= \int_0^T \big\langle p(w), Bu-Bw\big\rangle\,dt \\
&\quad= \big\langle B^*p(w), u-w\big\rangle_{\mathcal{U}} \\
&\quad= \big(\Lambda_{\mathcal{U}}^{-1}B^*p(w), u-w\big)_{\mathcal{U}}.
\end{aligned}$$

Hence (3.12) and (3.18) are equivalent, thus we obtained the characterization of the optimal control w. □

3.3 System with distributed control

Theorem 3.4 *Suppose $f \in L^2(0,T;H^{-1}(\Omega))$ and $\mathcal{U} = L^2(0,T;L^2(\Omega))$, then system* (1.2) *has a unique weak solution y such that*

$$y \in \begin{cases} H^\alpha(0,T;H_0^1(\Omega),H^{-1}(\Omega)), & 0 \le \alpha < \frac{1}{2}, \\ {}_0H^\alpha(0,T;H_0^1(\Omega),H^{-1}(\Omega)), & \frac{1}{2} < \alpha \le 1, \\ \{u \in H^{\frac{1}{2}}(0,T;H_0^1(\Omega),H^{-1}(\Omega)) : \int_0^T t^{-1}|y(t)|^2\,dt < \infty\}, & \alpha = \frac{1}{2}, \end{cases} \tag{3.23}$$

and for $\alpha \in (\frac{1}{2},1)$ we have $y \in C_u(0,T;L^2(\Omega))$ apart from a set of measure zero.

Proof We introduce the spaces

$$V = H_0^1(\Omega), \qquad H = L^2(\Omega), \qquad V' = H^{-1}(\Omega).$$

Hence we have

$$A(t)y = -\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial y}{\partial x_j}\right).$$

Take $\mathrm{Dom}(A(t)) = V = H_0^1(\Omega)$ and $A(t) \in \mathcal{L}(H_0^1(\Omega),H^{-1}(\Omega))$.

For $v \in V$ from Green's formula we have

$$\int_\Omega A(t)y(t)v\,dx = a\big(t;y(t),v\big), \tag{3.24}$$

hence we obtain

$$a\big(t;y(t),v\big) = -\sum_{i,j=1}^n \int_\Omega a_{ij}(x,t)\frac{\partial y}{\partial x_j}\frac{\partial v}{\partial x_i}\,dx,$$

and from (1.3), (1.4) we see that $a(t;y(t),v)$ satisfies condition (A1) to (A3).

Denote $f_1 = f + u \in L^2(0,T;H^{-1}(\Omega))$. From Definition 2.7, (1.1) is equivalent to the variational formulation

$$\begin{cases} \partial_t^\alpha (y(t),v) + a(t;y(t),v) = \langle f_1(t),v\rangle \quad \text{in } \mathcal{D}'(]0,T[), t\in(0,T], \\ \forall v\in H_0^1(\Omega), \\ y_0 = 0. \end{cases} \tag{3.25}$$

From Theorem 3.1 we see that there exists a unique solution y to (3.25) (and hence to system (1.2)) which satisfies (3.23). From Definition 2.8 this is also the unique weak solution to system (1.2). From Lemma 2.4 we obtain for $\alpha \in (\frac{1}{2},1)$, $y\in C_u(0,T;L^2(\Omega))$ apart from a set of measure zero. □

Finally we consider the optimal control of (1.2) with regard to the cost function:

$$J(u) = \int_0^T \int_\Omega \bigl(y(t;u) - z_d\bigr)^2\,dx\,dt + (Nu,u), \quad u\in\mathcal{U}_{ad}, z_d\in L^2(Q). \tag{3.26}$$

Now we give interpretations of notations from Section 3.2 in this situation. The observation space $\mathcal{H} = \mathcal{H}' = L^2(0,T;L^2(\Omega))$. C is the injection of $L^2(0,T;H_0^1(\Omega))$ onto $L^2(0,T;L^2(\Omega))$. B, Λ, $\Lambda_{\mathcal{U}}$ are identify mappings.

From (1.3) and (1.4) we can see that $A(t)$ is a self adjoint operator such that

$$A^* = -\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial y(w)}{\partial x_j}\right).$$

From Theorem 3.3 we can obtain a characterization of optimal control w of system (1.2) by simultaneously solving the following system of partial differential equations and inequality:

$$\begin{cases} \partial_t^\alpha y(w) - \sum_{i,j=1}^n \frac{\partial}{\partial x_i}(a_{ij}(x,t)\frac{\partial y(w)}{\partial x_j}) = f + w, \quad t\in(0,T], \\ y_0(w) = 0, \end{cases} \tag{3.27}$$

$$\begin{cases} {}_t^C D_T^\alpha p(w) - \sum_{i,j=1}^n \frac{\partial}{\partial x_i}(a_{ij}(x,t)\frac{\partial y(w)}{\partial x_j}) = y(w) - z_d, \quad t\in(0,T], \\ p(T;w) = 0, \end{cases} \tag{3.28}$$

$$\bigl(p(w) + Nw, u - w\bigr) \ge 0, \quad \forall u,w\in\mathcal{U}_{ad}, \tag{3.29}$$

$$\begin{cases} y(w)\in L^2(0,T;H_0^1(\Omega)), \\ p(w)\in L^2(0,T;H_0^1(\Omega)). \end{cases} \tag{3.30}$$

The symbol (,) in (3.29) denotes a scalar product in $L^2(0,T;L^2(\Omega))$.

In this work we considered fractional diffusion equation with Dirichlet boundary conditions with distributed control. It will be interesting also to consider a system with boundary controls or observations using the weak formulation given here.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions
Qing Tang read and approved the final manuscript. Qingxia Ma contributed during the revision of this article, read and approved of it.

Author details
[1]School of Mathematics and Physics, China University of Geosciences, Lumo 388, Wuhan, 430074, China. [2]Wuhan Institute of Physics and Mathematics, Chinese Academy of Sciences, Wuhan, 430071, China.

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